Lecture 17: CRHF & Merkle-Damgård Construction

Recall

- Collision-resistant Hash Function family from domain $\mathcal{D}$ to range $\mathcal{R}$ is a set of hash functions

$$\mathcal{H} = \{h^{(i)} \colon i \in \mathcal{I}\},$$

where $\mathcal{I}$ is the set of indices and each function $h^{(i)} \colon \mathcal{D} \to \mathcal{R}$

- Any efficient adversary given $h^{(i)}$, where $i \xleftarrow{\$} \mathcal{I}$, can output $x, x' \in \mathcal{D}$ such that $h^{(i)}(x) = h^{(i)}(x')$ only with negligible probability
- One bit compressing (i.e., $|\mathcal{D}| = 2\,|\mathcal{R}|$) can be constructed from the hardness of the discrete logarithm assumption as follows. Let the discrete logarithm problem be hard in the group G, then for $b \in \{0,1\}$ and $x \in \mathbb{Z}_{|G|}$, we have:

$$h^{(y)} \colon \{0,1\} \times \mathbb{Z}_{|G|} \to G$$

$$h^{(y)}(b,x) = y^b g^x$$

$$\mathcal{H} = \{h^{(y)} \colon y \in G\}$$

t-bit Compression

We can construct a t-bit compression function as follows: Let $b \in \{0,1\}^t$ and $y^{(1)}, \ldots, y^{(t)} \in \mathbb{Z}_{|G|}$.

$$h^{\left(y^{(1)},\ldots,y^{(t)}\right)}(b, x) = {y^{(1)}}^{b_1} \cdots {y^{(t)}}^{b_t} g^x$$

Each function is indexed by $(y^{(1)}, \ldots, y^{(t)})$ and each $y^{(i)} \in \{0,1\}^n$. So, index size is tn.

- Prove: If Discrete Logarithm assumption holds in G then the construction above is a CRHF, where $t = \mathrm{poly}(n)$
- Prove: If $\mathcal{H}^{(n)}$ is a CRHF family with functions $\{0,1\}^{n+1} \to \{0,1\}^n$, for all large enough n, then the construction above is a CRHF family, where $t = \mathrm{poly}(n)$
- Think: What is the difference between the above two theorems

Length-Halving CRHF

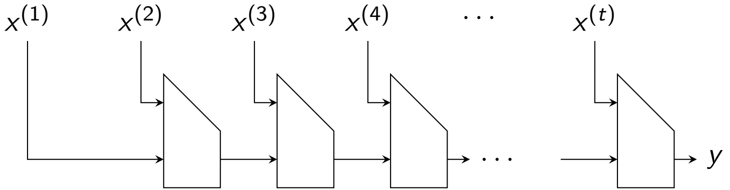
- In particular, with $t = n$ and $G = \{0,1\}^n$, the previously constructed function is a length halving family of hash functions where all functions are $\{0,1\}^{2n} \rightarrow \{0,1\}^n$

Tree-based Hashing

- We are interested in hashing $\{0,1\}^{tn}$ down to $\{0,1\}^{n}$
- One-bit compression at a time needs $(t-1)n \times n$ size indices. Can we do better?

Tree-based Hashing

- Let $\mathcal{H}$ be a CRHF family with functions $\{0,1\}^{2n} \to \{0,1\}^n$ and key size K
- We will construct CRHF family $\mathcal{H}^{(t)}$ with functions $\{0,1\}^{tn} \to \{0,1\}^n$ and key size K, for $t \geqslant 2$
- Let $x \in \{0,1\}^{tn}$ be represented as $(x^{(1)}, \dots, x^{(t)})$, where each $x^{(i)} \in \{0,1\}^n$. The function is calculated in an iterated fashion as represented below. Each box represents an application of a function $h \in \mathcal{H}$ and the output of the hash function is y. Call this new function $\mathrm{itr}_t(h)$ function. So, we have $\mathcal{H}^{(t)} = \{\mathrm{itr}_t(h) \colon h \in \mathcal{H}\}$.



Proof

- Our adversary $\widetilde{\mathcal{A}}$ on input a hash function h feeds $\text{itr}_t(h)$ function to $\mathcal{A}^*$
- Suppose $\mathcal{A}^*$ produces $x = (x^{(1)}, \dots, x^{(t)})$ and $z = (z^{(1)}, \dots, z^{(t)})$ such that it is a collision of the function $\text{itr}_t(h)$ function
- Suppose the input to the last h-box in the evaluation of $\text{itr}_t(h)(x)$ is a and the input to the last h-box in the evaluation of $\text{itr}_t(h)(z)$ is b. We know that the output of the last h-box is same in these two cases. If $a \neq b$, then we have found a collision.
- If $a = b$, then the output of the second last h-box is identical in $\text{itr}_t(h)(x)$ and $\text{itr}_t(h)(z)$ evaluation. Therefore, we can recurse on $(x^{(1)}, \dots, x^{(t-1)})$ and $(z^{(1)}, \dots, z^{(1)} t - 1)$ that also produce a collision (i.e. the output of the second last h-box)